The University of Chicago

SEMI-CONTINUITY IN THE CALCULUS OF VARIATIONS, AND ABSOLUTE MINIMA FOR ISO-PERIMETRIC PROBLEMS

A DISSERTATION SUBMITTED TO THE
GRADUATE FACULTY IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

By
EDWARD JAMES McSHANE

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CHICAGO, ILLINOIS
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This dissertation is one of a group of five submitted during the year 1930 to the Department of Mathematics of the University of Chicago and concerned with topics in the theory of the Calculus of Variations. The group as a whole has been published in book form by the University of Chicago Press under the title *Contributions to the Calculus of Variations, 1930.*

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INTRODUCTION

Tonelli, in his Fondamenti di Calcolo delle Variazione[1] has shown the considerable advantages in generality possessed by the method which he designates as the direct method in the Calculus of Variations. Many well-known theorems, notably those for the existence of the absolute minimum, have been obtained by Tonelli under hypotheses distinctly weaker than those presupposed by earlier writers. Prof. L. M. Graves has in a recent article[2] extended the existence theorems of the non-parametric problem to space of higher dimensions, with some simplification of the proofs.

In the present paper the existence theorems of the parametric problem and of the isoperimetric problem in parametric form, as well as a number of other important theorems due to Osgood, Lindeberg, and Tonelli are extended to n-dimensional space. In the case of the isoperimetric problem a single existence theorem has been demonstrated which includes all of Tonelli's. In the proofs a considerable simplification has

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[1] Leonida Tonelli, Fondamenti di Calcolo delle Variazione. The two volumes which have already appeared will be henceforward referred to as Tonelli I, Tonelli II.

[2] L. M. Graves, Annals of Mathematics, vol. 28 (1927) p. 153.

resulted from the use of a property of the Weierstrass E-function stated in our Lemma 1. From this lemma we are able to establish rather directly several theorems on semi-continuity, as well as the well-known theorems of Osgood and Lindeberg.

For the sake of brevity we shall assume the ordinary theory of Lebesgue integration, and use the Zermelo postulate freely.

PART I

NOTATIONS AND DEFINITIONS

In order to avoid excessive use of superscripts we shall use matrix notation. For example, z will stand for $(z^{(1)}, \cdots, z^{(n)})$, $f_x(x,y)$ for $(f_{x^{(1)}}(x^{(1)}, \cdots, x^{(n)}, y^{(1)}, \cdots y^{(n)}), f_{x^{(2)}}(x^{(1)}, \cdots x^{(n)}, y^{(1)}, \cdots y^{(n)}), \cdots, f_{x^{(n)}}(x^{(1)}, \cdots, y^{(n)}))$, yz for $y^{(1)}z^{(1)} + y^{(2)}z^{(2)} + \cdots + y^{(n)}z^{(n)}$, etc.[1] It will be assumed once and for all that the integrand functions $f(x,x')$, $g(x,x')$, etc., used in the sequel are continuous with their first and second partial derivatives, and positively homogeneous of degree 1 in x'; i.e., for every $k>0$, $f(x,kx') = kf(x,x')$. A will designate a point set in n-dimensional space, and we will assume always that $f(x,x')$, $g(x,x')$, etc., are defined for all x in A and for all x'.

By a continuous curve we shall understand a point set defined by a system of continuous functions $x(t)$ $(a \leq t \leq b)$. If $t_2 > t_1$, and not all the functions $x(t)$ are constant on the interval $[t_1, t_2]$, $x(t_2)$ is said to succeed $x(t_1)$. Two curves will be regarded as identical if and only if they consist of the same points in the same order, i.e., $x(t)$ $(a \leq t \leq b)$ is identical with $\xi(\tau)$ $(\alpha \leq \tau \leq \beta)$ if and only if there is a correspondence between t and τ such that $x(t) = \xi(\tau)$, and

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[1] For a more complete explanation of this notation see Bliss, Annals of Mathematics, vol. 6 (2nd series), 1905 p. 58 et seq.

$t_2 \geq t_1$ if and only if $\tau_2 \geq \tau_1$; that is, in the correspondence t is a monotonic increasing function of τ, and vice versa. The functions $t(\tau)$ and $\tau(t)$ are, however, not necessarily continuous or single valued.

As is well known, a curve $C: x = x(t)$ is rectifiable if and only if each $x^{(i)}(t)$ is of bounded total variation. A rectifiable curve can always be given an absolutely continuous parameterization. For to each t corresponds a length $s(t)$ measured along C from $x(0)$ to $x(t)$; hence we can express C as $y(s)$, the order being preserved since s increases monotonically with t. It is easily seen that the derivatives of each $y^{(i)}(s)$ are not greater than 1 in absolute value, hence each $y^{(i)}(s)$ is absolutely continuous.

In order to define the distance $\|C_1, C_2\|$ of two curves[1] let us consider a pair π of continuous parameterizations $x_1(t), x_2(t)$ $(0 \leq t \leq 1)$ of C_1, C_2 respectively. Let d_π be the least upper bound of $\|x_1(t) - x_2(t)\|$ for all t. (The symbol $\|y\|$ designates $\sqrt{yy}$.) $\|C_1, C_2\|$ is defined to be the greatest lower bound of d_π for all π. However, in the succeeding pages we shall wish to restrict ourselves to rectifiable curves and absolutely continuous parameterizations; hence we shall prove that if C_1 and C_2 be rectifiable, the value of $\|C_1, C_2\|$ is unaltered if only absolutely continuous parameterizations

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[1] For a detailed exposition of this subject, see Fréchet, Trans. Am. Math. Soc., vol. 6 (1905), p. 435.

are admitted.

We first show that if $x(t)$ $(0 \leq t \leq 1)$ be a continuous parameterization of a rectifiable curve C, then for every $\epsilon > 0$ there exists a parameterization $y(t)$ $(0 \leq t \leq 1)$ which is absolutely continuous (in fact, satisfies a Lipschitz condition) and is such that $\| x(t) - y(t) \| < \epsilon$ for every t. Let $X(s)$ be the parameterization of C by length of arc; $X(s(t)) = x(t)$. The curve $s = s(t)$ is continuous, for otherwise there would be an s and consequently an $X(s)$ with no corresponding value of t; therefore we can inscribe in it a polygon $s = \sigma(t)$ for which $|\sigma(t) - s(t)| < \epsilon$ for every t. $\sigma(t)$ is a monotonically increasing function of t, and $|\sigma(t_2) - \sigma(t_1)/t_2 - t_1| \leq M$, where M is the upper bound of the slopes of the sides of the polygon. Now if $y(t) \equiv X(\sigma(t))$, we have

$$\left| \frac{y^{(i)}(t_2) - y^{(i)}(t_1)}{t_2 - t_1} \right| = \left| \frac{X^{(i)}(\sigma(t_2)) - X^{(i)}(\sigma(t_1))}{\sigma(t_2) - \sigma(t_1)} \right| \left| \frac{\sigma(t_2) - \sigma(t_1)}{t_2 - t_1} \right| \leq 1 \cdot M,$$

so that $y(t)$ satisfies a Lipschitz condition and is absolutely continuous. Also, $\| y(t) - x(t) \| = \| X(\sigma(t)) - X(s(t)) \| \leq |\sigma(t) - s(t)| < \epsilon$, as was desired.

Letting $^*\| C_1, C_2 \|$ stand for the distance obtained by considering only absolutely continuous parameterizations, it is clear that $^*\| C_1, C_2 \| \geq \| C_1, C_2 \|$, since we are using only a sub-class of the class of continuous parameterizations. On the other hand, for every $\epsilon > 0$ we can find a continuous pair of parameterizations $x_1(t)$, $x_2(t)$ for which $\| x_1(t) - x_2(t) \| \leq$

$\|C_1, C_2\| + \epsilon/3$, and a pair of absolutely continuous parameterizations $y_1(t), y_2(t)$ for which $\|x_1(t) - y_1(t)\| \leq \epsilon/3$ and $\|x_2(t) - y_2(t)\| \leq \epsilon/3$; hence $\|y_1(t) - y_2(t)\| \leq \|C_1, C_2\| + \epsilon$, and $^*\|C_1, C_2\| \leq \|C_1, C_2\|$, proving their equality.

We define $\lim_{n\to\infty} C_n = C_0$ to mean $\lim_{n\to\infty} \|C_n, C_0\| = 0$. A curve C is in $(C_0)_\delta$ (read "in the δ -neighborhood of C_0 ") if $\|C, C_0\| < \delta$.

In order to avoid repetition we shall in the future understand that all curves mentioned are continuous and rectifiable and all parameterizations absolutely continuous unless mention is made to the contrary.

If C be a continuous rectifiable curve in A, defined by the absolutely continuous functions $x(t)$ $(a \leq t \leq b)$, then the integral

$$F(C) \equiv \int_C f(x, x')\,dt \equiv \int_a^b f(x(t), x'(t))\,dt$$

can be shown[1] to exist, and to be independent of the particular parameterization of C.

The integral $F(C)$ will be called:

positive definite, if $f(x, x') > 0$ for every x in A and every x' such that $x'x' > 0$;

positive semi-definite, if $f(x, x') \geq 0$ for the above mentioned arguments;

positive quasi-regular, if $E(x, x', \bar{x}') \geq 0$ for every x in A and

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[1] Tonelli I, p. 219.

every $x', \bar{x}'$ different from zero, where $E(x, x', \bar{x}') \equiv f(x, \bar{x}') - \bar{x}' f_{x'}(x, x')$;

positive quasi-regular semi-normal, if $F(C)$ is positive quasi-regular, and in addition $E(x, x', -x') > 0$ for every x in A and every $x' \neq 0$.

We note that $E(x, x', -x') = f(x, x') + f(x, -x')$. The last definition seems quite different from that of Tonelli[1], but by use of the formula of Schwarz[2] we see that in the two-dimensional case the two definitions are equivalent.

The letter K or some modification of it will designate a class of curves C, continuous, rectifiable and lying in A. K is called complete if every rectifiable limit curve of curves of K is itself a curve of K; of total ramification if for every C in K, every point P on C, and every continuous rectifiable arc PQ lying in A, the curve obtained by traversing C to P, PQ to Q, back to P, and thence along C, is again a curve of K. $K[F=h]$ is a symbol for the class of all curves of K for which $F(C) = h$.

The integral $F(C)$ is said to be lower semi-continuous at C_0 if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every continuous rectifiable curve C in $(C_0)_\delta$, $F(C) \geq F(C_0) - \epsilon$. This is equivalent to the condition that for every sequence $\{C_n\}$ such that $\lim_{n\to\infty} C_n = C_0$, $\lim_{n\to\infty} F(C_n) \geq F(C_0)$. If $F(C)$ is

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1) Tonelli I, page 224.

2) Tonelli I, p. 210; Bolza, Vorlesungen über variationsrechnung, p. 244.

lower semi-continuous at C for every continuous rectifiable curve C we say simply that $F(C)$ is lower semi-continuous.

The notion of lower semi-continuity is an essential one in the direct method of the Calculus of Variations. Continuity is too restrictive; it is a property only of certain integrals in the highly special form $F(C) \equiv \int_C M(x)x'dt$. On the other hand, semi-continuity is a property of almost all the classical examples of the Calculus of Variations; in fact, of all those admitting a strong relative minimum or maximum.

We shall often have occasion to restrict the argument x' in an integrand $f(x, x')$ to those vectors such that $x'x' = 1$. For brevity, we shall reserve the letter u for such vectors; in such an expression as $g(x, u')$ we understand always that $u'u' = 1$. We shall also use the symbol $\mathcal{L}(C)$ to denote the length of the curve C.

PART II

THE PARAMETRIC PROBLEM

Most of the following proofs depend upon the following fundamental

LEMMA 1. *Let the sequence* $\{C_n\}$ *have the limit curve* C_0 *and be so parameterized that* $\lim_{n\to\infty} x_n^{(i)}(t) = x_0^{(i)}(t)$ $(a \le t \le b)$, *for every* i, *uniformly, while the sequence* $\{x_n^{(i)\prime}(t)\}$ *of derivatives is uniformly bounded. Then*

$$\overline{\lim_{n\to\infty}}\,[F(C_n) - F(C_0)] = \overline{\lim_{n\to\infty}} \int_a^b E(x_0, x_0', x_n')\,dt,$$

$$\underline{\lim_{n\to\infty}}\,[F(C_n) - F(C_0)] = \underline{\lim_{n\to\infty}} \int_a^b E(x_0, x_0', x_n')\,dt,$$

where for all t *such that* $x_0'(t) = 0$ *we replace* x_0' *by any unit vector such that* x' *remains measurable. In particular, this will be true if we set* X', *a constant unit vector, in place of* x_0' *wherever the latter vanishes.*

To prove this we decompose the difference $F(C_n) - F(C_0)$ as follows:

$$\begin{aligned} F(C_n) - F(C_0) &= \int_a^b f(x_n, x_n')\,dt - \int_a^b f(x_0, x_0')\,dt \\ &= \int_a^b [f(x_n, x_n') - f(x_0, x_n')]\,dt \\ &\quad + \int_a^b [f(x_0, x_n') - x_n' f_{x'}(x_0, x_0')]\,dt \\ &\quad + \int_a^b (x_n' - x_0')\, f_{x'}(x_0, x_0')\,dt. \end{aligned}$$

In the last step we have used the well known relationship $x' f_{x'}(x, x') = f(x, x')$, consequence of the homogeneity of $f(x, x')$. Since $\lim_{n\to\infty} x_n = x_0$ uniformly, and $f(x, x')$ is continuous and the arguments bounded, the first integral approaches zero. Since $\lim_{n\to\infty} \int_a^t (x_n^{(i)\prime} - x_0^{(i)\prime})\,dt = \lim_{n\to\infty} [x_n^{(i)}(t) - x_0^{(i)}(t) + x_n^{(i)}(a) - x_0^{(i)}(a)] = 0$ uniformly,

we have[1] $\lim_{n\to\infty}\int_a^b(x_n'-x_o')f_{x'}(x_o,x_o')\,dt=0$. The remaining term of the right is $\int_a^b E(x_o,x_o',x_n')dt$, giving us the desired result.

COROLLARY. If, under the hypotheses of lemma 1, either one of $\lim_{n\to\infty}|F(C_n)-F(C_o)|$, $\lim_{n\to\infty}\int_a^b E(x_o,x_o',x_n')\,dt$ exist, so does the other, and the two are equal.

LEMMA 2. If $\lim_{n\to\infty}C_n=C_o$, and the sequence $\{C_n\}$ be of uniformly bounded length, then C_o and a subsequence of $\{C_n\}$ can be so parameterized as to satisfy the hypotheses of Lemma 1.

Let us choose as parameter for each C_n the ratio t of the arc length s_n (measured from the initial point) to the total length $L(C_n)$, which by hypothesis does not exceed a certain constant N. We thus obtain a sequence $\{x_n(t)\}$ $(0\le t\le 1)$ such that $|x_n^{(i)\prime}(t)|=\left|\frac{dx^{(i)}}{ds_n}\right|\cdot\left|\frac{ds_n}{dt}\right|\le 1\cdot L(C_n)\le N$. Since the $x_n^{(i)}(t)$ have uniformly bounded derivatives, by Ascoli's theorem[2] they have a limit function $x_o^{(1)}(t)$. Choose a subsequence of $\{C_n\}$ such that on this subsequence $\lim_{n\to\infty}x_n^{(1)}(t)=x_o^{(1)}(t)$ uniformly. In this subsequence, the $x_n^{(2)}(t)$ have a limit function $x_o^{(2)}(t)$. Choose again a subsequence on which $\lim_{n\to\infty}x_n^{(2)}(t)=x_o^{(2)}(t)$. Repeating this for each superscript, we obtain a subsequence $\{C_{n_m}\}$ of the sequence $\{C_n\}$ such that $\lim_{m\to\infty}x_{n_m}^{(i)}=x_o^{(i)}$ uniformly, for all values of t. Now the

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[1] Hobson, Theory of Functions of a Real Variable, vol. II, page 422.

[2] Tonelli I, p. 78.

curve $x_0(t)$ is a limit curve of $\{C_n\}$. But by hypothesis $\{C_n\}$ has the unique limit C_0, hence $x_0(t)$ is a particular parameterization of C_0, and the lemma is proved.

THEOREM I. *If $F(C)$ be positive quasi-regular, then for every number N, $F(C)$ is lower semi-continuous on the class of all curves of length $\leq N$.*

Suppose the theorem false. Then there exists a C_0, an $\epsilon > 0$, and a sequence $\{C_n\}$ of curves of length $\leq N$ such that $\lim_{n\to\infty} C_n = C_0$, and $F(C_n) \leq F(C_0) - \epsilon$ for every n. By Lemmas 2,1 we have on a subsequence of $\{C_n\}$ $\overline{\lim_{n\to\infty}} F(C_n) - F(C_0) = \overline{\lim_{n\to\infty}} \int_0^1 E(x_0, x_0', x_n')\,dt \geq 0$, whereas the relation $F(C_n) \leq F(C_0) - \epsilon$ implies that $\overline{\lim_{n\to\infty}} F(C_n) - F(C_0) \leq -\epsilon$.

LEMMA 3. *If A is bounded and closed, and K is a class of continuous rectifiable curves lying in A, and $F(C)$ is positive definite, then every minimizing sequence for $F(C)$ in K is of uniformly bounded length.*

By a minimizing sequence for $F(C)$ in K we mean a sequence $\{C_n\}$ of curves of K such that $\lim_{n\to\infty} F(C_n) = i$, where i is the greatest lower bound of $F(C)$ on K.

On the bounded closed set [x in A, x' such that $x'x' = 1$] the function $f(x, x')$ is continuous and > 0, hence there exists an $m > 0$ such that on this set $f(x, x') \geq m$. Letting L_n be the length of C_n, and using s as parameter on C_n, we have $F(C_n) = \int_0^{L_n} f(x, x')\,ds \geq \int_0^{L_n} m\,ds = m L_n$, so that $L_n \leq m^{-1} F(C_n)$. But on the sequence $\{C_n\}$, $\lim_{n\to\infty} F(C_n) = i$; hence $\{F(C_n)\}$ is uniformly bounded, and therefore so is $\{L_n\}$.

The hypothesis that $F(C)$ is positive definite is sufficient, but by no means necessary. Tonelli, in the beginning of his second volume, replaces it by weaker hypotheses. Most of Tonelli's results are covered by a theorem given by H.Hahn[1]. Hahn, working in two dimensions, replaces positive definiteness by the condition that $F(C)$ be positive semi-definite, positive quasi-regular, and the class of all curves C in A on which $F(C)=0$ be of uniformly bounded length.

THEOREM II. If A is bounded and closed, and K is a complete class of continuous rectifiable curves lying in A, and $F(C)$ is positive definite, positive quasi-regular, then there exists a minimizing curve for $F(C)$ in K.

Construct a minimizing sequence $\{C_n\}$ for $F(C)$ in K. By Lemma 3 $\{C_n\}$ is of uniformly bounded length.[2] Hence by Hilbert's theorem[3] there exists a limit curve C_0 of the sequence $\{C_n\}$, which belongs to K since K is complete. By Th.I $F(C_0) \leq \lim_{n\to\infty} F(C_n) = i$. But since C_0 is in K, $F(C_0) \geq i$; comparing the inequalities, $F(C_0) = i$, and C_0 is the curve sought.

Let us introduce the notation (O,R) to designate the hypersphere whose center is the origin of coordinates and whose

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[1] Wiener Sitzungsberichte, Abteilung IIa, 134 Band (1925), p. 437.

[2] Since this is the only use we make of the hypothesis of positive definiteness, the theorem could be improved for two dimensional space by substituting the condition of Hahn mentioned above.

[3] Tonelli I, p. 87.

radius is R. In order to treat the case in which the set A is unbounded, we prove

LEMMA 4. *If K is a class of curves every one of which has at least one point interior to a certain (O,R), and $F(C)$ is positive semi-definite, and there exists a $\mu > 0$ such that $|f(x,u')| \geq \mu/\|x\|$ for all x exterior to (O,R) and all u', then for every $N > i$ we can find an $R_1 > R$ such that the class $K[F \leq N]$ lies in (O,R_1).*

Let R_1 be a number greater than $Re^{N/\mu}$. Every curve C having a point Q exterior to (O,R_1) has an arc $C(P,Q)$ with P on (O,R) and otherwise exterior to (O,R), and of length $\geq R_1 - R > Re^{N/\mu} - R$. Indicating by s the length of arc measured along C from P, we have

$$\begin{aligned} F(C) \geq F(C(P,Q)) &\geq \mu \int_{C(P,Q)} \frac{ds}{\|x\|} \geq \mu \int_{C(P,Q)} \frac{ds}{R+s} \\ &\geq \mu \left[\log (R+s)\right]_0^{R_1-R} = \mu \left[\log R_1 - \log R\right] \\ &> \mu \left[\log Re^{N/\mu} - \log R\right] = N. \end{aligned}$$

THEOREM III.

If (a) the part of A lying on (O,Q) is closed for every positive Q;

(b) K is a complete set of curves lying in A;

(c) $F(C)$ is positive definite, positive quasi-regular;

(d) the hypotheses of Lemma 4 are satisfied;

then there exists a minimizing curve for $F(C)$ in K.

Choose $N > i$, and let R_1 be a number large enough so that $K[F \leq N]$ lies in (O,R_1); that such a number exists

we know by lemma 4. The problem of minimizing $F(C)$ in K is clearly equivalent to that of minimizing $F(C)$ in $K[F \leq N]$, and since this class lies in the bounded closed set common to A and $(0, R_1)$, by Th. II a minimizing curve exists.

PART III

THE ISOPERIMETRIC PROBLEM

The following theorem, although a special case of Th. V, is so easily proved as to justify its separate statement:

THEOREM IV. *If* $G(C) \equiv \int_C M(x)x'dt$; *where the* $M(x)$ *are continuous on* A, *and* $F(C) \equiv \int_C f(x,x')\,dt$ *be positive definite, positive quasi-regular, and* A *be bounded and closed, and* K *a complete class of curves in* A, *then, for every* g_o *there exists a minimizing curve for* $F(C)$ *in* $K[G = g_o]$.

Here and henceforward it will be assumed that the class $K[G = g_o]$ referred to is not a vacuous class.

Denoting by f_o the greatest lower bound of $F(C)$ on $K[G = g_o]$, we choose a sequence $\{C_n\}$ of curves of $K[G = g_o]$ for which $\lim_{n\to\infty} F(C_n) = f_o$. By lemma 3 this sequence is of uniformly bounded length, hence by Hilbert's theorem it has a limit-curve C_o. Applying lemmas 2, 1 and the corollary to lemma 1, we see that on a subsequence of $\{C_n\}$

$$g_o - G(C_o) = \lim_{n\to\infty} [G(C_n) - G(C_o)] = \lim_{n\to\infty} \int_{t_o}^{t} E_g(x_o, x_o', x_n')\,dt = 0;$$

hence C_o is in $K[G = g_o]$, and $F(C_o) \geq f_o$. But by Th. I $F(C_o) \leq f_o$, hence $F(C_o) = f_o$, and C_o is the curve sought.

Before proceeding to the main theorem, we will introduce some new notations. We shall be concerned with three

integrals, $F(C)$, $G(C)$ and $H(C) \equiv \int_C h(x, x')\,dt$, the latter being positive quasi-regular. Let C be a rectifiable curve $x(s)$, where s is the arc length. As is well known, $x'(s)$ exists and $\| x'(s) \| = 1$ except possibly on a set M of measure 0. If now there exist numbers γ, Γ such that

1) $\gamma E_h(x(s), x'(s), \bar{x}') \leq E_g(x(s), x'(s), \bar{x}') \leq \Gamma E_h(x(s), x'(s), \bar{x}')$

for all s not in M and for all $\bar{x}'$, we define $\gamma(C)$ to be the least upper bound of all such γ, and $\Gamma(C)$ to be the greatest lower bound of all such Γ. $\varphi(C)$ and $\Phi(C)$ are the corresponding bounds when g is replaced by f, and $\gamma^*(C), \Gamma^*(C)$, $\varphi^*(C), \Phi^*(C)$ are the corresponding bounds when $\bar{x}'$ is restricted to be equal to $-x'$. It is easily seen that if there exists an $s, \bar{x}'$ such that $E_g(x(s), x'(s), \bar{x}') \neq 0$ while $E_h(x(s), x'(s), \bar{x}') = 0$, at least one of $\gamma(C), \Gamma(C)$ is undefined; a similar remark applies to the $\varphi(C), \Phi(C)$. Also, if defined, the $\gamma(C), \Gamma(C), \varphi(C), \Phi(C)$ are infinite if and only if $E_h(x(s), x'(s), \bar{x}') = 0$ for all s and all $\bar{x}'$.

We will use combinations of the following hypotheses:

H_1) <u>There exist an $\alpha \geq 0$ and a β such that $\alpha F(C) + \beta G(C)$ is positive definite on A.</u>

H_2) <u>A is bounded and closed, and K is a complete class of curves of total ramification lying in A; or, if A is unbounded, the hypotheses of Lemma 4 are satisfied by $\alpha F(C) + \beta G(C)$.</u>

H_3) <u>There exist numbers $a \geq 0$, b, not both zero, such that $aF(C) + bG(C)$ is positive quasi-regular.</u>

We observe that if H_3 be satisfied, we may consider that at least one of $F(C)$, $G(C)$ is positive quasi-regular. For either $G(C)$ is quasi-regular, and if negative quasi-regular may be replaced by $-G(C)$; or $a > 0$, and the problem of minimizing $F(C)$ in $K[G=g_0]$ is equivalent to that of minimizing $aF(C)+bG(C)$ in $K[G=g_0]$, and this is by H_3 positive quasi-regular.

H_4) There exists an integral $H(C) \equiv \int_C h(x,x')dt$ which is positive quasi-regular, and constants Γ, γ, φ such that $\varphi E_h(x,u';\bar{u}') \leqq E_f(x,u';\bar{u}')$, $\gamma E_h(x,u';\bar{u}') \leqq E_g(x,u';\bar{u}') \leqq \Gamma E_h(x,u';\bar{u}')$ for all x in A, $\bar{u}' \neq u'$.

We notice that if H_4 be fulfilled $\Gamma(C)$, $\gamma(C)$, $\Gamma^*(C)$, $\gamma^*(C)$, $\varphi(C)$, $\varphi^*(C)$ are defined for every curve C. This is not necessarily true of $\Phi(C)$, $\Phi^*(C)$.

H_5) For every curve C of K either

(a) $\varphi(C) \geq 0$ and $\Gamma(C) \leqq 0$, or

(b) we can find an arc σ_1 lying in A and intersecting C on which $\Phi^*(\sigma)$ is defined and $\Phi^*(\sigma_1) \leqq \varphi(C)$, $\gamma^*(\sigma_1) \geq \Gamma(C)$, while on a subset of σ_1 of positive measure[1] $E_g(x,u';-u') \neq 0$, where u' gives the direction of σ_1.

H_6) For every curve C of K either

(a) $\varphi(C) \geq 0$ and $\gamma(C) \geqq 0$, or

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[1] For this it is sufficient that $E_g(x,u';-u')$ be not identically zero on σ_1. The proof of this is based on the theorem of §403, Hobson, The Theory of Functions of a Real Variable, vol I.

(b) there exists an arc σ_2 lying in A and intersecting C, on which $\Phi^*(\sigma_2)$ is defined and $\Phi^*(\sigma_2) \leq \varphi(C)$, $\Gamma^*(\sigma_2) \leq \gamma(C)$, while on a subset of σ_2 of positive measure $E_g(x, u', -u') \neq 0$, where u' gives the direction of σ_2.

Since the hypotheses H_1 - H_6 are more stringent than necessary for the existence of a minimizing curve, it is of some interest to find necessary conditions which limit the possible weakening of hypotheses. To do this, we show that if A is a closed region and K the class of all curves of A joining two closed point sets in A, then in order that for every g_0 there shall exist a minimizing curve for $F(C)$ in $K[G = g_0]$ it is necessary that there exist an $\alpha \geq 0$ and a β such that $\alpha E_f(x, x', -x') + \beta E_g(x, x', -x') \geq 0$ for all x in A, $x' \neq 0$. We prove this for the interior of A; by continuity it continues to hold true on the boundary. If there exist a k, an (x_1, x_1') and an (x_2, x_2') such that $E_g(x_1, x_1', -x_1') > 0$, $E_g(x_2, x_2', -x_2') < 0$, $E_f(x_1, x_1', -x_1') < k E_g(x_1, x_1', -x_1')$, $E_f(x_2, x_2', -x_2') < k E_g(x_2, x_2', -x_2')$, let us construct a curve of the class K through x_1 and x_2. Let σ_1, σ_2 be line segments through x_1, x_2 in directions x_1', x_2' short enough so that along them the above inequalities remain true. Let Σ_1 be an arc formed by traversing and retracing σ_1 a finite number of times; $2G(\Sigma_1) = \int_{\Sigma_1} E_g(x, x', -x')\,dt > 0$. We can find a Σ_2 formed similarly from σ_2 such that $G(\Sigma_2) = -G(\Sigma_1)$. Then $F(\Sigma_1) + F(\Sigma_2) < k[G(\Sigma_1) + G(\Sigma_2)] = 0$. Then the curve C_n formed by adding n

repetitions each of Σ_1, Σ_2 to C has $G(C_n) = G(C)$, while $\lim_{n\to\infty} F(C_n) = -\infty$.

Hence if for every g_0 the minimizing curve is to exist, we must have either: for no (x, x') is $E_g(x, x', -x') > 0$ (in which case $\alpha = 0$, $\beta = -1$ serve); or for no (x, x') is $E_g(x, x', -x') < 0$ (in which case $\alpha = 0$, $\beta = 1$ serve), or else for every k and every (x_1, x_1'), (x_2, x_2') such that $E_g(x_1, x_1', -x_1') > 0 > E_g(x_2, x_2', -x_2')$ either $E_f(x_1, x_1', -x_1') \geq k E_g(x_1, x_1', -x_1')$ or

In this last case we consider a fixed (x_1, x_1') and let be the lease upper bound of all k_0 such that $E_f(x_1, x_1', -x_1') \geqq k_0 E_g(x_1, x_1', -x_1')$ Then for all $\epsilon > 0$ it is false that $E_f(x_1, x_1', -x_1') \geq (k + \epsilon) E_g(x_1, x_1', -x_1')$, so that for all (x_2, x_2') for which $E_g(x_2, x_2', -x_2') < 0$ we must have $E_f(x_2, x_2', -x_2') \geq (k + e) E_g(x_2, x_2', -x_2')$, whence $E_f(x_2, x_2', -x_2') \geq k E_g(x_2, x_2', -x_2')$. Reversing the argument and holding (x_2, x_2') fixed, we see that k serves for all (x, x') for which $E_g(x, x', -x') \neq 0$. Moreover, wherever $E_g(x, x', -x') = 0$, $E_f(x, x', -x')$ must be ≥ 0. For otherwise we construct through x a line segment σ of length l small enough so that on σ the inequality remains true, and denote by $\sigma(e)$ the portion of σ having x as end point and length e. Either $\int_{\sigma(e)} E_g(x, x', -x')\, dt$ vanishes for some $e_1 > 0$, in which case we let Σ_n be the curve obtained by traversing and retracing $\sigma(e_1)$ n times, or else it does not change sign on σ. In the latter case, let Σ_1 be formed by traversing and retracing σ once, and

Σ_n $(n>1)$ by traversing and retracing portions of $\sigma(l/n)$ enough times to make $\int_{\Sigma_n} E_g(x, x', -x')\,dt = \int_{\Sigma_1} E_g(x, x', -x')\,dt$. Letting C be any curve of K through x, and C_n the curve formed by adjoining Σ_n to C, we see that in either case $G(C_n)$ is constant, while $\lim_{n\to\infty} F(C_n) = -\infty$. Hence even when E_g vanishes $E_f + kE_g \geqq 0$, and our statement is proved.

In the important special case in which $f(x, -x') = f(x, x')$ and $g(x, -x') = g(x, x')$, the above condition is equivalent to requiring $\alpha F(C) + \beta G(C)$ to be positive semi-definite. For the purposes of our proof we need more than this, but on the other hand H is stronger than necessary. Further remarks concerning this will be found after Lemma 5.

By Weierstrass' necessary condition, if $C: x = x(s)$ be a minimizing curve for $F(C)$ in a class $K[G = g_0]$ there exist an $a \geqq 0$ and a b such that $a E_f(x(s), x'(s), \bar{x}') + b E_g(x(s), x'(s), \bar{x}') \geqq 0$ for all $\bar{x}'$. Moreover, we have seen above that in order that a minimizing curve exist for each g_0, it is necessary that there exist an $\alpha \geqq 0$, β for which $\alpha E_f(x, x', -x') + \beta E_g(x, x', -x') \geqq 0$ for all x, x'. These two conditions together seem to indicate that H_3 cannot be greatly weakened.

LEMMA 5. *If the system A, K, f, g satisfies H_1 and H_2, there exists for each g_0 a minimizing sequence for $F(C)$ in $K[G = g_0]$ which is bounded and of*

uniformly bounded length.

Since $\alpha \geq 0$, every minimizing sequence for $F(C)$ in $K[G = g_0]$ is also a minimizing sequence for $\alpha F(C) + \beta G(C)$ in that class. If the set A be bounded, so is the sequence, and by Lemma 3 it is bounded in length. If A be unbounded, by Lemma 4 every minimizing sequence lies except possibly for a finite number of its terms in a certain sphere (O, R_1), and this case reduces to the one just considered.

Remark 1. If the conclusion of Lemma 5 is valid, then the lower bound f_0 of $F(C)$ in $K[G = g_0]$ is finite for each g_0. For, letting $\{C_n\}$ be the minimizing sequence mentioned and M the least upper bound of $|f(x, u')|$ for all in A and all u', $|F(C_n)| \leq ML(C_n)$ hence the $F(C_n)$ are a bounded set and their limit f_0 is finite.

Remark 2. H_1 is more than is needed to insure the existence of a minimizing sequence of the type mentioned. The conclusion of Lemma 5 would still be valid if there existed an $\alpha \geq 0$ and a β such that $\alpha F(C) + \beta G(C)$ satisfies the condition of Hahn, for then every minimizing sequence for $\alpha F + \beta G$ (including those for $F(C)$ alone) in the class $K[G = g_0]$ is of uniformly bounded length. The fact that $K[G = g_0]$ is not necessarily complete does not affect the conclusion as to the lengths. Moreover, if we can take $\alpha > 0$, then every minimizing sequence for $\alpha F + \beta G$ is also a minimizing sequence for $F(C)$ in $K[G = g_0]$, and

a condition similar to that of Tonelli II, pages 49-67 could be substituted for H_1, but the proof would be quite tedious.

THEOREM V. If the system A, K, f, g be such that the conclusion of Lemma 5 is valid, and also H_3, H_4, H_5, H_6 are fulfilled, then for every g_0 there exists a minimizing curve for $F(C)$ in $K[G = g_0]$.

Let $\{C_n\}$ be the minimizing sequence of the conclusion of Lemma 5. By Hilbert's theorem $\{C_n\}$ has a limit curve C_0. Choose a subsequence of $\{C_n\}$ having the unique limit C_0 and parameterize it (by lemma 2) so as to satisfy the hypotheses of Lemma 1. On this subsequence the numbers $H(C_n)$ are a bounded set and hence have a limit point h_0. We choose a subsequence on which $\lim_{n\to\infty} H(C_n) = h_0$, and in the sequel restrict ourselves to this subsequence.

Let us adopt temporarily the notation $B(x, x', \bar{x}')$ to designate $E_g(x, x', \bar{x}') - \gamma(C_0) E_h(x, x', \bar{x}')$, agreeing to replace $\gamma(C_0)$ by an arbitrary finite number in case $\gamma(C_0)$ is infinite (which, as we have seen, cannot occur unless $E_h(x_0(s), x_0'(s), \bar{x}') = 0$ for all s and all $\bar{x}'$). We shall use $y_0(s)$ to designate the parameterization of C_0 by arc length, and $x_0(t)$ for the parameterization introduced by the process of Lemma 2. Since $B(x, u', \bar{u}')$ is uniformly continuous for all x in a neighborhood of C_0 and for all u' and $\bar{u}'$, and $y_0(s)$ is continuous in s, it follows that to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that if $|s_1 - s_2| < \delta$

(2) $|B(y_0(s_1), u', \bar{u}') - B(y_0(s_2), u', \bar{u}')| < \epsilon$. Let us now subdivide

the interval $0 \leq s \leq \mathcal{L}(C_0)$ into subintervals $[\alpha_i, \beta_i]$ of length less than δ; to these correspond t-intervals $[\tau_i, \sigma_i]$, where, for definiteness, in case an interval of t's corresponds to an end-point α_i we choose as τ_i the left end-point of the interval. In each interval $[\tau_i, \sigma_i]$ we choose a t_i such that $y_0'(s(t_i))$ exists and $\|y_0'(s(t_i))\| = 1$. By the definition of $\gamma(C_0)$, $B(y_0(s(t_i)), y_0'(s(t_i)), \bar{u}') \geq 0$ for each i. (In case $\gamma(C_0)$ is infinite, by H_4, $B(y(s), y'(s), \bar{x}') \equiv 0$, and this and succeeding inequalities are still valid.) Also, for every t in $[\tau_i, \sigma_i]$ we have $|s(t) - s(t_i)| < \delta$, so that by the inequality (2), $B(y_0(s(t)), y_0'(s(t_i)), \bar{u}') > -\epsilon$.

Now define $\eta_\epsilon(t)$ as equal to $y_0'(s(t_i))$ on all of $[\tau_i, \sigma_i]$; $\eta_\epsilon(t)$ being constant by stretches is measurable, and the preceding inequality becomes $B(x_0(t), \eta_\epsilon(t), \bar{u}') > -\epsilon$, which is valid for all t and all $\bar{u}'$. In order now to apply lemma 1, we first note that $\|x_n'(t)\|$ has an upper bound, say U, for all t and all n; and we define $X'(t)$ as equal to $x_0'(t)$ where $x_0'(t)$ exists and is not zero, and equal to $\eta_\epsilon(t)$ elsewhere. Then by Lemma I

$$\begin{aligned}[g_0 - G(C_0)] - \gamma(C_0)[h_0 - H(C_0)] &= \lim_{n\to\infty} \int_0^1 E_g(x_0, X', x_n')\,dt - \lim_{n\to\infty} \int_0^1 E_h(x_0, X', x_n')\,dt \cdot \gamma(C_0) \\ &= \lim_{n\to\infty} \int_0^1 B(x_0, X', x_n')\|x_n'\|\,dt \geq -\epsilon U;\end{aligned}$$

and since the left side is independent of ϵ, it must be positive or zero.

A similar argument is applicable to expressions analogous to B, but with $\gamma(C_0)$ replaced by $\Gamma(C_0)$ or by

$\varphi(C_0)$, so that we finally obtain:

(3) $f_0 - F(C_0) \geq \varphi(C_0)[h_0 - H(C_0)]$,

(4) $\Gamma(C_0)[h_0 - H(C_0)] \geq g_0 - G(C_0) \geq \gamma(C_0)[h_0 - H(C_0)]$.

Suppose that Σ is a curve generated by starting from C_0 along the σ_1 of H_5 or the σ_2 of H_6 and retracing our path a finite number of times. Since on Σ we have $\Phi^*(\Sigma) \leq \varphi(C_0)$,

$$2F(\Sigma) = 2\int_\Sigma f(x,x')\,ds = \int_\Sigma [f(x,x') + f(x,-x')]\,ds$$
$$= \int_\Sigma E_f(x,x',-x')\,ds \leq \varphi(C_0)\int_\Sigma E_h(x,x',-x')\,ds$$
$$= 2\varphi(C_0)H(\Sigma),$$

which with (3) proves that

(5) $F(C_0+\Sigma) = F(C_0) + F(\Sigma) \leq f_0 - \varphi(C_0)[h_0 - H(C_0) - H(\Sigma)]$.

We are now ready to consider the separate cases that may arise.

Case 1. $H(C_0) = h_0$.

By (4), $g_0 = G(C_0)$, so that C_0 is in $K[G = g_0]$ and $F(C_0) \geq f_0$. But by (3) $f_0 \geq F(C_0)$, so that $F(C_0) = f_0$, and C_0 is the curve sought.

Case 2. $\varphi(C_0) \geq 0$, $g_0 = G(C_0)$.

By (3), $F(C_0) \leq f_0$. But since $G(C_0) = g_0$, we must have $F(C_0) \geq f_0$, so that C_0 is the curve sought.

Case 3. $\varphi(C_0) \geq 0$, $g_0 > G(C_0)$.

Using H_5b) and equation (2), we see that on a Σ_1 formed by traversing σ_1 and retracing, $G(\Sigma_1) > 0$, and by repeating often enough we can make $G(\Sigma_1) = g_0 - G(C_0)$ Then

(6) $\Gamma(C_0)[h_0 - H(C_0)] \geq g_0 - G(C_0) = G(\Sigma_1) = \frac{1}{2}\int_{\Sigma_1} E_g(x,x'-x)dt$
$\geq \frac{1}{2}\int_{\Sigma_1}\Gamma(C_0)\,E_h(x,x'-x)\,dt = \Gamma(C_0)\cdot H(\Sigma_1)$,

so that $h_0 - H(C_0) \geq H(\Sigma_1)$, and by (5) $F(C_0+\Sigma_1) \leq f_0$. But since $G(C_0+\Sigma_1) = g_0$, $F(C_0+\Sigma_1) \geq f_0$. Hence $F(C_0+\Sigma_1) = f_0$, and $C_0+\Sigma_1$ is the curve sought.

Case 4. $\varphi(C_0) \geq 0$, $g_0 < G(C_0)$.

Equation (4) shows that $\gamma(C_0) < 0$, so that H_6 b) is valid. On σ_2 of H_6 b) $\Gamma^*(\sigma_2) < 0$, hence $E_g(x,x'-x') \leqq 0$ and on a set of positive measure is < 0. We then construct a Σ_2 analogous to Σ_1 on which $G(\Sigma_2) = g_0 - G(C_0)$. Proceeding as with (6);

$G(\Sigma_2) \leqq \gamma(C_0)\cdot H(\Sigma)$; hence $h_0 - H(C_0) \geq H(\Sigma_2)$, and by (5) $F(C_0+\Sigma_2) \leq f_0$. Reasoning as before, $C_0+\Sigma_2$ is the curve sought.

Case 5. $\varphi(C_0) < 0$, $g_0 = G(C_0)$.

Since by the remark after H_3 we may assume one of E, G to be positive quasi-regular, and F clearly is not, we may assume $E_g \geqq 0$. H_6 b) holds in this case. Since it is assumed that on σ_2 $E_g(x,x'-x')$ does not vanish identically, $\Gamma^*(\sigma_2) > 0$, and $\gamma(C_0) \geq \Gamma^*(\sigma_2) > 0$. Hence by (4) $h_0 - H(C_0) = 0$, which we have already seen implies that C_0 is the minimizing curve sought.

Case 6. $\varphi(C_0) < 0$, $g_0 > G(C_0)$.

H_6 b) is fulfilled; use σ_2 to form a Σ_2 on which $G(\Sigma_2) = g_0 - G(C_0)$. Then $\gamma(C_0)[h_0 - H(C_0)] \leqq g_0 - G(C_0)$ $= G(\Sigma_2) \leq \gamma(C_0)H(\Sigma_2)$. Since $g_0 - G(C_0) > 0$ the last

inequality shows $\gamma(C_0) > 0$, so that $h_0 - H(C_0) \leq H(\Sigma_2)$. Then by the inequality (5) $F(C_0 + \Sigma_2) \leq I_0$, and by the usual argument $C_0 + \Sigma_2$ is the curve sought.

Case 7. $\varphi(C_0) < 0$, $g_0 < G(C_0)$.

This case cannot arise; for by inequality (4) it requires that neither F nor G be positive quasi-regular, contrary to the remark after H_3.

For greater ease in application to specific problems, we state several corollaries of Th. V. These corollaries are extensions to n-space of the theorems in Tonelli II, pages 466-482. Definition. We will say, with Tonelli, that the function $\psi(x)$ satisfies in A the condition α (condition β) if for every point x_0 of A through which passes at least one continuous rectifiable arc in A it is always possible to conduct a continuous rectifiable arc lying in A and without multiple points, on which $\psi(x) \leq \psi(x_0)$ $(\psi(x) \geq \psi(x_0))$.

As a first corollary we have Theorem IV of this paper. To show that it is included under Th. V, take $\alpha = a = 1$, $\beta = b = 0$, $h(x, x') \equiv f(x, x')$. Then in H_4 $\varphi = 1$, $\Gamma = \gamma = 0$. For every C, $\varphi(C) = 1$, $\Gamma(C) = \gamma(C) = 0$. H_1, H_2, H_3, H_4, $H_5 a)$ and $H_6 a)$ are satisfied.

COROLLARY 2. If $G(C)$ be positive quasi-regular, positive definite; if $F(C) = \int_C [\psi(x) g(x, x') + M(x) x'] dt$, where the function $\psi(x)$ is continuous and satisfies cond. α and the n functions $M(x)$ are continuous; if A be bounded and closed, and K a complete class of curves of total rami-

fication; then for every g_0 there exists a minimizing curve for $F(C)$ in $K[G = g_0]$.

Take $\alpha = a = 0$, $\beta = b = 1$, $h(x,x') = g(x,x')$ In H_4, $\Gamma = \gamma = 1$, $\varphi =$ lower bound of $\psi(x)$ on A. The arc provided by cond. α serves both as the σ_1 of H_5 b) and as the σ_2 of H_6 b). $E_g(x, x'-x')$ does not vanish on σ_1, σ_2 or in fact on A, being equal to $g(x,x') + g(x,-x') > 0$.

COROLLARY 3. If $F(C) \equiv \int_C f(x,x')\,dt$ be positive quasi-regular, positive definite;
if $G(C) = \int_C [\psi(x) f(x,x') + M(x)x']dt$, where the $M(x)$ are continuous, and the function $\psi(x)$ is continuous and satisfies cond. β wherever $\psi(x) > 0$ and cond. α wherever $\psi(x) < 0$;
if A be bounded and closed, and K a complete class of curves of total ramification in A;
then for every g_0 there exists a minimizing curve for $F(C)$ in $K[G = g_0]$.

Take $\alpha = a = 1$, $\beta = b = 0$, $h(x,x') \equiv f(x,x')$. H_1, H_2, H_3, H_4 are fulfilled. For each curve C of K there exists an on for which $\psi(x_1) = \max \psi(x)$ on C, which is not less than $\Gamma(C)$, which is the upper bound of $E_g/E_h \equiv \psi(x)$ taken over the subset for which the denominator is not null. Either $\Gamma(C) \leq 0$, satisfying H_3 a), or through x_1 there passes by cond. β an arc on which $\psi(x) \geq \Gamma(C)$; this serves as σ_1 of H_5 b), since $\varphi(C) = \Phi(C) = \varphi^*(C) = \Phi^*(C)$ / for every curve C, and on σ_1
$E_g(x, x'-x') \geq \Gamma(C) E_f(x, x'-x') = \Gamma(C)[f(x,x') + f(x,-x')] > 0$.

COROLLARY 4. If $F(C)$ and $G(C)$ are both positive

quasi-regular, and $G(C)$ positive definite,

if A be bounded and closed,

if K be a complete class of curves of total ramification in A, and such that through each curve of K there passes a continuous rectifiable arc in A on which $f(x,x') = 0$ for all x on the arc and all x',

then for every q_0 there exists a minimizing curve for $F(C)$ in $K[G = q_0]$.

Take $\alpha = a = 0$, $\beta = b = 1$, $h(x,x') \equiv g(x,x')$. H_1, H_2, H_3, H_6 a) are clearly satisfied. In H_4, $\Gamma = \gamma = 1$, $\varphi = 0$ will serve. On the curve on which $f(x,x') = 0$ for all x', $E_f \equiv 0$; hence this curve serves as the σ_1 of H_5 b).

Examples:

Tonelli[1] has given a number of examples of isoperimetric problems satisfying the hypotheses of his theorems, and hence also of Corollaries 1-4. Bolza[2] has given a list of thirty-five problems, not all however concerned with the existence of a minimizing curve and not all in parametric form. Many of these are covered by Th. V; Probs. 1,2,3,12,30,29 are examples under Cor. 2, Prob. 10 under Corollary 1; Prob. 5 (to find equilibrium position of thread on frictionless surface, holding ends fixed) comes under Corollary 2 if the surface has no proper minimum, or in fact if there is no proper

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[1] II, chapter X.

[2] Vorlesungen über Variationsrechnung, pp. 528-541.

minimum in the portion of the surface reachable by the thread. Examples not covered by the theorem here stated are Prob. 6 (in which H_1 cannot be fulfilled), and in fact any problem in which both integrands vanish simultaneously.

An example under Th. V but not under any corollary or theorem of Tonelli is that of minimizing $\int_C \sqrt{x'^2 + (1+x^2)y'^2}\,dt$ in the class of all curves of given length joining $(-1,0)$ and $(1,0)$. H_1, H_2, H_3, H_4, H_5b), H_6a) are satisfied, using in H_5b) a segment of the y-axis for σ_1.

PART IV

MISCELLANEOUS THEOREMS

LEMMA 6.[1] If the sequence $\{C_n\}$ of continuous rectifiable curves has a continuous limit curve C_o (not necessarily rectifiable), and $F(C)$ is positive quasi-regular semi-normal on the set of points constituting C_o, and if $\lim_{n\to\infty} \mathcal{L}(C_n) = +\infty$, then $\lim_{n\to\infty} F(C_n)$ is also equal to $+\infty$.

By the remark on page 5 it is possible to choose parameterizations $x_n(t)$, $x_o(t)$ $(0 \le t \le 1)$ of the curves C_n, C_o respectively, such that $\lim_{n\to\infty} x_n(t) = x_o(t)$ uniformly, while the $x_n(t)$ are absolutely continuous. Denote by u' (or some modification of it) a unit vector; $\|u'\| = 1$. Then by hypothesis, for every u' and for every x on C_o we have $E(x, u', -u') > 0$; since the set $[x \text{ on } C, u']$ is bounded and closed and $E(x, u', -u')$ continuous on them, the lower bound is attained, and $E(x, u', -u') \ge 4m > 0$; and since $E(x, u', -u')$ is uniformly continuous, we can find a $\delta > 0$ such that if $\|u_1' - u_2'\| \le \delta$, $E(x, u_1', -u_2') \ge 2m$. We now subdivide the surface of the unit sphere into a finite

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[1] Lemmas 6 and 7 are equivalent to the theorems of Tonelli I pp. 326, 334, except that we do not require the limit curve C_o to be rectifiable nor the integral $F(C)$ to be positive quasi-regular semi-normal at points not on C_o.

number N of mutually exclusive portions Z_k small enough so that if u_1' and u_2' are in the same Z_k, $\|u_1' - u_2'\| \leq \delta$.

By the uniform continuity of $f(x, x')$ we can find a positive γ small enough so that if $\|x_1 - x_2\| \leq \gamma$, $|f(x_1, u') - f(x_2, u')| \leq m/N$. Since $x_0(t)$ is continuous, we can subdivide the interval $[0,1]$ into a finite number of subintervals Δ_j, such that if t_1 and t_2 are in the same Δ_j, $\|x_0(t_1) - x_0(t_2)\| \leq \gamma/2$; and we can find an n_0 such that for $n \geq n_0$, $\|x_n(t) - x_0(t)\| \leq \gamma/2$ for all t. Hence if $\bar{x}_j$ denote the point $x_0(t)$ corresponding to a value of t in Δ_j (say the mid-point of Δ_j), then for every t in Δ_j and for every $n \geq n_0$ we have $\|x_n(t) - \bar{x}_j\| \leq \gamma$, and hence $|f(x_n(t), u') - f(\bar{x}_j, u')| \leq m/N$.

Denote by $C_{n,j}$ the curve $x_n(t)$ $(t \text{ in } \Delta_j)$ and by $T_{j,k,n}$ the subset of Δ_j for which $u_n'(t) \equiv x_n'(t)/\|x_n'(t)\|$ is in Z_k. Then

$$\mathcal{L}(C_{n,j}) = \int_{\Delta_j} \|x_n'(t)\|\,dt = \sum_k \int_{T_{j,k,n}} \|x_n'(t)\|\,dt,$$

so that for at least one value of k

$$\int_{T_{j,k,n}} \|x_n'(t)\|\,dt \geq N^{-1}\mathcal{L}(C_{n,j}).$$

Consider $j = 1$. In Δ_1, for each n there exists a k such that the above inequality holds. But there are only N values of k and an infinitude of n's; hence at least one k, which we shall designate by $k(1)$, must serve for an infinitude of n's. Select the subsequence of $\{C_n\}$ corresponding to these n's. On this subsequence, consider $j = 2$, and repeat the above argument; after a finite number of repeti-

tions, we obtain a subsequence of $\{C_n\}$ and a set of $k(j)$, one for each Δ_j, for all of which

$$\int_{T_{j,k(j),n}} \|x_n'(t)\|\,dt \geq N^{-1}\mathcal{L}(C_{n,j}).$$

Denoting by $-\bar{u}_j'$ a vector arbitrarily chosen from $Z_{k(j)}$, we have

$$\begin{aligned} F(C_n) &= \sum_j \int_{\Delta_j} f(x_n, x_n')\,dt \\ &= \sum_j \Big\{ \int_{\Delta_j} [f(x_n, x_n') - f(\bar{x}_j, x_n')]\,dt \\ &\quad + \int_{\Delta_j} [f(\bar{x}_j, x_n') - x_n' f_{x'}(\bar{x}_j, \bar{u}_j')]\,dt \\ &\quad + \int_{\Delta_j} x_n' f_{x'}(\bar{x}_j, \bar{u}_j')\,dt \Big\}. \end{aligned}$$

The third integral in the brace equals

$$f_{x'}(\bar{x}_j, \bar{u}_j') \int_{\Delta_j} x_n'\,dt = f_{x'}(\bar{x}_j, \bar{u}_j')[x_n(\beta_j) - x_n(\alpha_j)],$$

where α_j and β_j are the end-points of Δ_j. This is bounded for all n. The absolute value of the first integral does not exceed

$$\int_{\Delta_j} \|x_n'(t)\| \cdot |f(x_n, u_n') - f(\bar{x}, u_n')|\,dt \leq \frac{m}{N} \int_{\Delta_j} \|x_n'(t)\|\,dt = \frac{m}{N}\mathcal{L}(C_n);$$

so that the sum as to j does not exceed $(m/N)\mathcal{L}(C_n)$ in absolute value. For the second integral,

$$\begin{aligned} &\sum_j \int_{\Delta_j} [f(\bar{x}_j, x_n') - x_n' f_{x'}(\bar{x}_j, \bar{u}_j')]\,dt \\ &= \sum_j \Big\{ \sum_k \int_{T_{j,k,n}} E(\bar{x}_j, \bar{u}_j', x_n')\,dt \Big\} = \sum_j \Big\{ \sum_k \int_{T_{j,k,n}} E(\bar{x}_j, \bar{u}_j', u_n'(t))\|x_n'\|\,dt \\ &\geq \sum_j \Big\{ \int_{T_{j,k(j),n}} E(\bar{x}_j, \bar{u}_j', u_n'(t))\,\|x_n'(t)\|\,dt \Big\} \\ &\geq 2m \sum_j \int_{T_{j,k(j),n}} \|x_n'(t)\|\,dt \geq \sum_j 2mN^{-1}\mathcal{L}(C_{nj}) = 2mN^{-1}\mathcal{L}(C_n). \end{aligned}$$

Hence $F(C_n)$ is, except for a bounded quantity, in excess of $mN^{-1}\mathcal{L}(C_n)$, and so tends to $+\infty$.

We have thus shown that if $\lim_{n\to\infty} \mathcal{L}(C_n) = +\infty$, then

for a certain subsequence of $\{C_n\}$ we also have $\lim_{n\to\infty} F(C_n) = +\infty$. But this must then be true on the whole sequence; for if $\lim_{n\to\infty} F(C_n) < \infty$, we could select a subsequence such that $\lim_{n\to\infty} F(C_n) = \lambda < \infty$, and by the above proof we could find a subsequence of the latter for which $\lim_{n\to\infty} F(C_n) = \infty$, which is absurd.

LEMMA 7. *If the sequence $\{C_n\}$ of continuous rectifiable curves approaches the continuous limit curve C_0; and if to every x_0 of C_0 there corresponds a unit vector u_0' such that $E(x_0, u_0', \bar{u}') > 0$ for all $\bar{u}' \neq u_0'$; and if $\lim_{n\to\infty} L(C_n) = +\infty$, then $\lim_{n\to\infty} F(C_n)$ is also equal to $+\infty$.*

Consider a point x_0 of C_0 and the corresponding u_0'. By hypothesis, $f(x_0, \bar{u}') - \bar{u}' f_{x'}(x_0, u_0') > 0$ for all $\bar{u}' \neq u_0'$; whence on the bounded closed set of $\bar{u}'$ such that $\|\bar{u}' - u_0'\| \geq 1$, $f(x_0, \bar{u}') - \bar{u}' f_{x'}(x_0, u_0') \geq 2\mu > 0$. The linear expression $\mu u_0' \bar{u}'$ is positive for all $\bar{u}'$ such that $\|\bar{u}' - u_0'\| \leq 1$, and for all $\bar{u}'$ is at most equal to μ in absolute value; hence if we define $M(x_0) \equiv f_{x'}(x_0, u_0') - \mu u_0'$, we have $f(x_0, \bar{u}') - M(x_0)\bar{u}' > 0$ for all $\bar{u}'$. By the continuity of $f(x, u')$ we can find a ρ such that if $\|x - x_0\| \leq 3\rho$, $f(x, \bar{u}') - M(x_0)\bar{u}' > 0$. By the Heine-Borel theorem a finite number of the neighborhoods $\|x - x_0\| \leq \rho$ cover the point set C_0. Letting ρ_0 be the smallest of the ρ, we divide the interval of definition $[0, 1]$ of the curves into a finite number of intervals $\Delta_j \equiv [\alpha_j, \beta_j]$ such that on each Δ_j the

oscillation of $x_0(t)$ does not exceed ρ_0. The point $x_0(\alpha_j)$ belongs to one of the ρ-neighborhoods; we will re-name the mid-point and linear function belonging to this ρ-neighborhood $\bar{x}_j, M_j$ respectively. It is clear that for every t in Δ_j, $x(t)$ is in $(\bar{x}_j)_{2\rho}$

We now decompose

$$F(C_n) = \sum_j \left\{ \int_{\Delta_j} [f(x_n, x_n') - x_n' M_j]\, dt \right\} + \int_{\Delta_j} M_j x_n'\, dt.$$

Since the M_j are a finite set of constants, the last integral is easily seen to be bounded. On each Δ_j, for large enough n $C_n(\Delta_j)$ lies in $(C_0(\Delta_j))_\rho$, which is in $(\bar{x}_j)_{3\rho}$; hence $f(x_n, u_n') - u_n' M_j \geq m_j > 0$. Let the smallest of the m_j be m; then

$$\sum_j \int_{\Delta_j} [f(x_n, x_n') - x_n' M_j]\, dt = \sum_j \int_{\Delta_j} \| x_n' \| [f(x_n, u_n') - u_n' M_j]\, dt$$
$$\geq m \sum_j \int_{\Delta_j} \| x_n' \|\, dt = m\, \ell(C_n),$$

so that $\lim_{n\to\infty} F(C_n) = +\infty$.

THEOREM VI.[1] <u>If C_0 be a continuous rectifiable curve, and $F(C)$ be positive quasi-regular semi-normal on the set of points constituting C_0, then $F(C)$ is lower semi-continuous at C_0.</u>

Suppose the theorem false. Then there exists a sequence $\{C_n\}$ such that $\lim_{n\to\infty} C_n = C_0$, $\lim_{n\to\infty} F(C_n) = \lambda < F(C_0)$. By Lemma 6 it is impossible that

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[1] Tonelli I, p. 288.

$\lim_{n\to\infty} \mathcal{L}(C_n) = \infty$, for $F(C_n)$ is bounded; hence the sequence contains a subsequence of uniformly bounded length, on which by theorem I $\lim_{n\to\infty} F(C_n) \geq F(C_0)$. This contradicts $\lim_{n\to\infty} F(C_n) = \lambda < F(C_0)$ and establishes the theorem.

COROLLARY 1.[1] If $F(C)$ be positive quasi-regular semi-normal on the whole field A it is lower semi-continuous on A.

COROLLARY 2. If $F(C)$ be positive quasi-regular on the point set constituting the continuous rectifiable curve C_0, and to each x_0 of C_0 corresponds a direction u_0' such that $E(x_0, u_0', \bar{u}') > 0$ for all $\bar{u}' \neq u_0'$, then $F(C)$ is lower semi-continuous on C_0.

This can be established in the same manner as theorem VI, using lemma 7 instead of lemma 6, but it is in fact a special case of theorem VI. For, for every x_0 and $\bar{u}'$ it is true that $0 < E(x_0, u_0', \bar{u}') + E(x_0, u_0', -\bar{u}') = f(x_0, \bar{u}') + f(x_0, -\bar{u}')$; and this, with the quasi-regularity of F, implies its semi-normality.

COROLLARY 3. If F be positive quasi-regular on A, and to each x of A corresponds a u_0' such that $E(x, u_0', \bar{u}') > 0$ for all $\bar{u}' \neq u_0'$, then $F(C)$ is lower semi-continuous on A.

We will use the notation σ to denote the ratio of the length of arc, from the beginning of a curve to a generic

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[1] Tonelli I, p. 269.

point on it, to the total length of the curve. Then $C = x(\sigma)$ $(0 \leq \sigma \leq 1)$ and it is clear that the $x(\sigma)$ are absolutely continuous, and that $\|x'(\sigma)\| = \mathcal{L}(C)$ almost everywhere.

THEOREM VII.[1] If the sequence $\{C_n\}$ approaches C_0 as limit, and $\lim_{n\to\infty} \mathcal{L}(C_n) = \mathcal{L}(C_0) > 0$, then for every $\delta > 0$, $\epsilon > 0$ we can find an $n_{\delta\epsilon}$ such that if $n \geq n_{\delta\epsilon}$ the angle between $x_n'(\sigma)$ and $x_0'(\sigma)$ is numerically less than ϵ except possibly on a set of measure less than δ.

We first show that if we choose σ as parameter on each C_n, $\lim_{n\to\infty} \|x_n(\sigma) - x_0(\sigma)\| = 0$ uniformly. Suppose that for some σ_1 it were false that $\lim_{n\to\infty} \|x_n(\sigma_1) - x_0(\sigma_1)\| = 0$; the $x_n(\sigma_1)$ would then have some other limit point, necessarily on C_0, say $x_0(\sigma_2)$. Choose a subsequence of $\{C_n\}$ on which $\lim_{n\to\infty} x_n(\sigma_1) = x_0(\sigma_2)$. Assume for definiteness $\sigma_2 > \sigma_1$, and denote by $\overline{C}_n$ the curve $x_n(\sigma)$ $(0 \leq \sigma \leq \sigma_1)$ and by $\overline{C}_0$ the curve $x_0(\sigma)$ $(0 \leq \sigma \leq \sigma_2)$. Now $\lim_{n\to\infty} \overline{C}_n = \overline{C}_0$, while $\lim_{n\to\infty} \mathcal{L}(\overline{C}_n) = \sigma_1 \mathcal{L}(C_0) < \sigma_2 \mathcal{L}(C_0) = \mathcal{L}(\overline{C}_0)$, which is impossible by the lower semi-continuity of $\mathcal{L}$. Hence $\lim_{n\to\infty} \|x_n(\sigma) - x_0(\sigma)\| = 0$ for every σ. To prove the uniformity we note that the lengths of $\{C_n\}$ are bounded, say $\leq M$, so that for all i, n, σ_1, σ_2 we have

$$\left| \frac{x_n^{(i)}(\sigma_2) - x_n^{(i)}(\sigma_1)}{\sigma_2 - \sigma_1} \right| \leq M.$$

For an arbitrary $\epsilon > 0$ we can divide the interval $[0, 1]$

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[1] Tonelli I, p. 95-105.

into equal sub-intervals of length not greater than $\epsilon/3M$, and choose n_ϵ large enough so that for the end-points σ_k of these intervals $|x_n^{(i)}(\sigma_k) - x_o^{(i)}(\sigma_k)| \leq \epsilon/3$ for all $n \geq n_\epsilon$; it follows readily that $|x_n^{(i)}(\sigma) - x_o^{(i)}(\sigma)| \leq \epsilon$ for all σ.

The set for which the angle $\alpha_n(\sigma)$ between $x_n'(\sigma)$ and $x_o'(\sigma)$ (that is, $arc\cos[x_n'x_o'/\|x_n'\|\cdot\|x_o'\|]$) is at least equal to ϵ is the same set as that on which $1 - \cos\alpha_n(\sigma) \geq 1 - \cos\epsilon$, and it is sufficient to show that $\lim_{n\to\infty}\int_0^1 (1 - \cos\alpha_n(\sigma))\,d\sigma = 0$. By lemma 1,

$$0 = \lim_{n\to\infty}[\mathcal{L}(C_n) - \mathcal{L}(C_o)] = \lim_{n\to\infty}\int_0^1 E(x_o, x_o', x_n')\,d\sigma$$
$$= \lim_{n\to\infty}\int_0^1 \|x_n'\|(1 - \cos\alpha_n(\sigma))\,d\sigma = \lim_{n\to\infty}\mathcal{L}(C_n)\int_0^1 (1-\cos\alpha_n(\sigma))\,d\sigma;$$

and since $\lim_{n\to\infty}\mathcal{L}(C_n) = \mathcal{L}(C_o) \neq 0$, the theorem is proved.

THEOREM VIII.[1] <u>If the sequence $\{C_n\}$ approaches C_o as a limit, and $\lim_{n\to\infty}\mathcal{L}(C_n) = \mathcal{L}(C_o)$, and $F(C)$ satisfies the usual hypotheses of continuity and homogeneity, then</u> $\lim_{n\to\infty} F(C_n) = F(C_o)$

It has been shown by Bliss[2] that $E_f(x, x', \bar{x}') = \|\bar{x}'\|(1 - \cos\alpha)[u_1' f_{x'x'}(x, u_2')\,u_1']$, where $\alpha = arc\cos(x'\bar{x}'/\|x'\|\cdot\|\bar{x}'\|)$ and u_1' and u_2' are suitably chosen unit vectors; the expression $|u_1' f_{x'x'}(x, u_2')\,u_1'|$ being bounded (say $\leq M$) for all x in A and for all u_1' and u_2', we have $E_f(x, x', \bar{x}') \leq M E_l(x, x', \bar{x}')$ for all $x, x', \bar{x}'$.

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[1] Tonelli I, p. 227.

[2] Trans. Am. Math. Soc., Vol. 15 (1914) p. 377.

Now if it were false that $\lim_{n\to\infty} F(C_n) = F(C_0)$, we could find a subsequence on which $\lim_{n\to\infty} |F(C_n) - F(C_0)| = \delta > 0$. But by lemmas 2, 1, we can find a subsequence of this on which

$$\lim_{n\to\infty} |F(C_n) - F(C_0)| = \lim_{n\to\infty} \left|\int_0^1 E_f(x_0, x_0', x_n')\,dt\right|$$
$$\leq M \lim_{n\to\infty} \left|\int_0^1 E_l(x_0, x_0', x_n')\,dt\right| = M \lim_{n\to\infty} |\mathcal{L}(C_n) - \mathcal{L}(C_0)| = 0,$$

which is a contradiction.

THEOREM IX.[1] *If C_0 is continuous and $F(C)$ is positive quasi-regular semi-normal on the point-set constituting C_0, then, for every number Φ we can find two positive numbers ρ, Λ such that for every C in $(C_0)_\rho$ for which $F(C) \leq \Phi$, it is true that $\mathcal{L}(C) < \Lambda$.*

Suppose the theorem false. Then we can find a sequence of pairs $\{\rho_n, \Lambda_n\}$ such that $\lim_{n\to\infty} \rho_n = 0$, $\lim_{n\to\infty} \Lambda_n = \infty$, and a sequence $\{C_n\}$ such that C_n is in $(C_0)_{\rho_n}$, $\mathcal{L}(C_n) \geq \Lambda_n$ and $F(C_n) \leq \Phi$. Then $\lim_{n\to\infty} C_n = C_0$, $\lim_{n\to\infty} \mathcal{L}(C_n) = \infty$, and $\overline{\lim_{n\to\infty}} F(C_n) \leq \Phi < \infty$, contrary to lemma 6.

THEOREM X. Theorem of Osgood generalized.[2] *If $F(C)$ be positive quasi-regular semi-normal; if C_0 be the unique minimizing curve for $F(C)$ in a given complete class K of curves C; then there exists a $\rho > 0$, such that to every positive $\rho_1 < \rho$ there corresponds a $\mu > 0$ having the property that for every curve C of K lying in $(C_0)_\rho$*

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[1] Tonelli I, p. 326.

[2] Tonelli II, p. 210; O. Bolza, Vorlesungen über variations rechning, p. 280; Osgood, Trans. Am. Math. Soc., Vol. II (1901), p. 273.

and having at least one point not interior to $(C_0)_{\rho_2}$, $F(C) - F(C_0) > \mu$.

In theorem IX, take $\Phi = F(C_0) + 1$; then there exists a ρ and a Λ such that every curve C in $(C_0)_{\rho_2}$ with $F(C) \leq F(C_0) + 1$ has length $\mathcal{L}(C) \leq \Lambda$. Now for every positive $\rho_1 < \rho$, the class K of curves of K lying in $(C_0)_\rho$, having at least one point not interior to $(C_0)_{\rho_2}$, and satisfying $F(C) \leq F(C_0)+1$ is by theorem I a complete class of curves of uniformly bounded length; hence, as in the proof of theorem II, there exists a minimizing curve C_1 for $F(C)$ in K_1. Since C_1 has at least one point not in $(C_0)_{\rho_2}$, it is distinct from C_0, hence $F(C_1) > F(C_0)$. Let $\mu = \frac{1}{2}(F(C_1) - F(C_0))$; then for every curve C of K_1, $F(C) - F(C_0) \geq F(C_1) - F(C_0) > \mu > 0$, which was to be proved.

THEOREM XI[1] If C_0 be continuous and rectifiable, and $F(C)$ be such that

1) to every x_0 on C_0 corresponds a u_*' such that $E(x_0, u_*', \bar{u}') > 0$ for every $\bar{u}' \neq u_*'$;

2) for almost every s for which $x_0'(s)$ is defined $E(x_0, x_0', \bar{u}') \geq 0$ for all $\bar{u}'$;

then $F(C)$ is lower semi-continuous at C_0.

Assume the theorem false. Then there exists an $\epsilon > 0$ and a sequence $\{C_n\}$ such that $\lim_{n\to\infty} C_n = C_0$, $\lim_{n\to\infty} F(C_n) =$

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[1] Tonelli I, p. 288.

$F(C_0) - \epsilon$. By lemma 7, this sequence has uniformly bounded lengths, say $\leq L$. Parameterize by lemma 2 in such a way as to make lemma 1 applicable. In order to define the x_0' of lemma 1 at the points where $x_0'(t)$ fails to exist or vanishes, we note that by the continuity of $E(x, u', \bar{u}')$, to each t_0 there corresponds a $\delta > 0$ such that if $|t - t_0| < \delta$, $E(x_0(t), u_*'(t_0), \bar{u}') \geq -\epsilon/2L$, where $\bar{u}'$ is any unit vector and $u_*'(t_0)$ is the u_*' corresponding to $x_0(t_0)$. By the Heine-Borel theorem a finite number of such intervals cover the interval $[0,1]$; defining $U'(t)$ to be equal in each interval to the corresponding $u_*'(t_0)$, we obtain a function constant by stretches. Now let $u_0'(t) \equiv x_0'(t)/\|x_0'(t)\|$ where $x_0'(t)$ exists and is not zero, and $u_0'(t) \equiv U'(t)$ elsewhere; $u_0'(t)$ is measurable, and for all values of t it is easily seen that $E(x_0(t), u_0'(t), \bar{u}') \geq -\epsilon/2L$.

Now using lemma 1,

$$\lim_{n\to\infty} F(C_n) - F(C_0) = \lim_{n\to\infty} \int_0^1 E(x_0(t), u_0'(t), x_n'(t))\,dt \geq L(-\epsilon/2L) = -\epsilon/2,$$

contradicting the previous result that $\lim_{n\to\infty} F(C_n) - F(C_0) - \epsilon$ and establishing the theorem.

THEOREM XII. Theorem of Lindeberg generalized.[1] If $F(C)$ is such that on the continuous rectifiable curve C_0: a)

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[1] Tonelli I, p. 321.
Tonelli, Rend. R. Accad. dei Lincei, 1921, 1° sem., p. 19. J. W. Lindeberg, Math. Annalen, vol. 67 (1909) p. 340.

to each x_0 of C_0 there corresponds a $u_*'(x_0)$ such that $E(x_0, u_*', \bar{u}) > 0$ for $\bar{u}' \neq u_*'$; b) u_*' can be taken equal to $x_0'(s)/\|x_0'(s)\|$ whenever the ratio is defined; then for every $\delta > 0$ we can find $\epsilon > 0$, $\rho > 0$ such that for every C in $(C_0)_\rho$ for which $L(C) - L(C_0) \geq \delta$, it is true that $F(C) - F(C_0) \geq \epsilon$.

Otherwise for every pair of infinitesimal sequences $\{\rho_n\}$, $\{\epsilon_n\}$ we could find a sequence $\{C_n\}$ such that for every n C_n is in $(C_0)_{\rho_n}$, $F(C_n) - F(C_0) \leq \epsilon$, while $L(C_n) - L(C_0) \geq \delta$. It is impossible by Lemma 7 that $\lim_{n\to\infty} L(C_n) = \infty$, for $F(C_n)$ is bounded. Parameterize, by Lemma 2, a subsequence of this so as to make Lemma 1 applicable and choose a subsequence for which $L(C_n)$ has a definite limit L. With this parameterization, we designate by M the set of values of t for which $x_0'(t)$ vanishes or is undefined, and distinguish two cases:

Case 1. The set M has measure 0.

On the complement of M, $u_0'(t) \equiv x_0'(t)/\|x_0'(t)\|$ is the ratio of non-vanishing measurable functions, hence measurable; on M define $u_0'(t) \equiv u_*'(x_0(t))$; the function $u_0'(t)$, and hence $E'(x_0(t), u_0'(t), x_n'(t))$, is measurable on $[0,1]$. By Lemma 1, $\lim_{n\to\infty} \int_0^1 E_1(x_0, x_0', x_n')\,dt = \lim_{n\to\infty} \int_0^1 (1 - \cos\alpha_n)\|x_n'(t)\|\,dt \geq \delta$ Hence there exists an $\eta > 0$ such that on a subset S_{1n} of $[0,1]$ of measure at least 4η, $\cos\alpha_n \leq 1 - 4\eta$. S_{1n} contains[1]

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[1] Carathéodory, Vorlesungen über Reele Funktionentheorie, p. 275, satz 3.

a closed subset S_{2n} of measure $\geq 3\eta$. Since $u_0'(t)$ is measurable, we can find by Lusin's theorem[1] a subset S_3 of $[0,1]$ with measure $\geq 1-\eta$ on which $u_0'(t)$ is continuous; and we can find a closed subset S_4 of S_3, with measure $\geq 1-2\eta$. The set $S_{5n} \equiv S_{2n} \cdot S_4$ is closed and has measure $\geq \eta$, and on it $u_0'(t)$ is continuous. On the bounded closed point set $[x_0(t), u_0'(t)\ (t \text{ in } S_3);\ \bar{u}'$ such that $\cos\alpha \leq 1-4\eta]$ $E(x_0(t), u_0'(t), \bar{u}') > 0$, hence $\geq m > 0$. Hence by lemma 1

$$\lim_{n\to\infty} F(C_n) - F(C_0) = \lim_{n\to\infty} \int_0^1 E_t(x_0(t), u_0'(t), x_n'(t))\,dt$$
$$= \lim_{n\to\infty} \int_0^1 E_t(x_0, u_0', u_n')\,\|x_n'\|\,dt = \lim_{n\to\infty} \mathcal{L}(C_n) \int_0^1 E_t(x_0, u_0', u_n')\,dt$$
$$\geq \lim_{n\to\infty} \mathcal{L}(C_n) \int_{S_{5n}} E_t(x_0, u_0', u_n')\,dt \geq L\,m\eta > 0.$$

But $F(C_n) - F(C_0) \leq \epsilon_n$ for every n by the first paragraph of this proof; this is the desired contradiction.

Case 2. The set M has positive measure.

We define, as usual, the statement that M has metric density 1 at a point t_0 to mean that for every positive ϵ there corresponds a δ_ϵ such that for all intervals I of length not greater than δ_ϵ which contain t_0 the difference $1 - (mI \cdot M / mI)$ does not exceed ϵ; it is known[2] that every measurable set has metric density 1 at almost all its points. Choose a point t_0 of M of metric density 1, and let the ϵ be $\frac{1}{6}$; then for every h not greater than a certain

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[1] N. Lusin, Comptes Rendus de l'Acad. des Sc., 154, (1912) p. 168; W. Sierpiński, Fund. Math. III (1922) p. 320.

[2] Hobson, Theory of Functions of a Real Variable, vol. I, p. 181.

δ ($= \frac{1}{2}\delta_\epsilon$) the measure of the subset M_h of M lying on the interval $I_h \equiv [t_0 - h, t_0 + h]$ is not less than $5/6 \cdot 2h = \frac{5}{3}h$. (Modifications needed if t_0 is on end point of $[0, 1]$ are obvious.) On M_h, define $u_0'(t) \equiv u_*'(x_0(t_0))$; on the remainder of the interval I_h, $u_0'(t) \equiv x_0'(t)/\|x_0'(t)\|$; this function is clearly measurable. By $[\mathcal{L}_n]_{t_0-h}^{t_0+h}$ we shall mean the length of the arc of C_n corresponding to the interval I_h. Since $\|x_0'(t)\| = 0$ almost everywhere in M, and nowhere exceeds L, $[\mathcal{L}_0]_{t_0-h}^{t_0+h} = \int_{t_0-h}^{t_0+h} \|x_0'(t)\|\,dt \leq \frac{1}{3}hL$.

Recalling the method of parameterizing in Lemma 2, $[\mathcal{L}_n]_{t_0-h}^{t_0+h} = 2h\mathcal{L}(C_n)$; so that

$$2hL - \tfrac{1}{3}hL \leq \lim_{n\to\infty} [\mathcal{L}_n - \mathcal{L}_0]_{t_0-h}^{t_0+h} = \lim_{n\to\infty} \int_{t_0-h}^{t_0+h} (1 - \cos\alpha_n)\,\mathcal{L}_n\,dt$$
$$= L \lim_{n\to\infty} \int_{t_0-h}^{t_0+h} (1 - \cos\alpha_n)\,dt,$$

whence $\lim_{n\to\infty} \int_{t_0-h}^{t_0+h} \cos\alpha_n\,dt \leq h/3$.

By an easy calculation we find that for all sufficiently large n the measure of the set A_n of points of $[t_0 - h, t_0 + h]$ at which $\cos\alpha_n \leq \frac{3}{4}$ must be at least $h/2$.

For all $\bar{u}'$ for which $\cos(u_*'(x_0(t_0)), \bar{u}') \leq \frac{3}{4}$, $E(x_0(t_0), u_*'(x_0(t_0)), \bar{u}')$ is positive, say $\geq 19\mu > 0$; and we can find a $\gamma > 0$ such that for all t in $[t_0 - \gamma, t_0 + \gamma]$ we have $|E(x_0(t), u_*'(x_0(t_0)), \bar{u}') - E(x_0(t_0), u_*'(x_0(t_0)), \bar{u}')| \leq \mu$. Choosing h as the smaller of δ, γ, we have for all t in MA_n $E(x_0(t), u_*'(x_0(t_0)), u_n'(t)) \geq 18\mu$, and for all t in $M(I_h - A_n)$ $E(x_0(t), u_*'(x_0(t_0)), u_n'(t)) \geq -\mu$.

Combining these results, and using Lemma 1,

$$\begin{aligned}\lim_{n\to\infty} F(C_n) - F(C_0) \geq\ & \lim_{n\to\infty} [F(C_n) - F(C_0)]_0^{t_0-h} \\ &+ \lim_{n\to\infty} [F(C_n) - F(C_0)]_{t_0+h}^{1} \\ &+ \lim_{n\to\infty} \int_{I_h - M} E(x_0(t), x_0'(t), x_n'(t))\, dt \\ &+ \lim_{n\to\infty} \int_{MA_n} E(x_0(t), x_*'(x_0(t_0)), x_n'(t))\, dt \\ &+ \lim_{n\to\infty} \int_{M(I_h - A_n)} E(x_0(t), x_*'(x_0(t_0)), x_n'(t))\, dt.\end{aligned}$$

Of the five terms on the right, the first two are positive by Theorem XI and the third because the integrand is non-negative. The measure of MA_n is at least $h/6$ and the integrand of the fourth integral is at least $18\mu\mathcal{L}_n$, so that its lower limit is not less than $3\mu h L$. The measure of $M(I_h - A_n)$ is at most $3h/2$ and the integrand not less than $-\mu\mathcal{L}_n$, so that its lower limit is not less than $-\frac{3}{2}\mu h L$. Combining, $\lim_{n\to\infty} F(C_n) - F(C_0) \geq \frac{3}{2}\mu h L$. But by the original choice of the $\{C_n\}$ we had $F(C_n) - F(C_0) \leq \epsilon_n$ for every n, which is a contradiction.

THEOREM XIII.[1] If $\lim_{n\to\infty} C_n = C_0$, and $F(C)$ is such that to every x_0 on C_0

1) there corresponds a x_*' such that $E_f(x_0, x_*', \bar{x}') > 0$ for all $\bar{x}' \neq x_*'$;

2) whenever $\frac{dx_0}{ds}$ is defined, x_*' can be taken equal to $\frac{dx_0}{ds}$;

if $G(C)$ satisfies the usual conditions of continuity and homogeneity;

if $\lim_{n\to\infty} F(C_n) = F(C_0)$; then

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[1] Tonelli I, p. 331, 332.

A) $\lim_{n\to\infty} \mathcal{L}(C_n) = \mathcal{L}(C_0)$;

B) $\lim_{n\to\infty} G(C_n) = G(C_0)$.

By Theorem XII, for every $\delta > 0$ we can find $\epsilon > 0, \rho > 0$ such that for C_n in $(C_0)_\rho$ and $F(C_n) - F(C_0) < \epsilon$, $\mathcal{L}(C_n) < \mathcal{L}(C_0) + \delta$. Both conditions are fulfilled for all n sufficiently large; hence $\overline{\lim_{n\to\infty}} \mathcal{L}(C_n) \leq \mathcal{L}(C_0)$. But by the lower semi-continuity of $\mathcal{L}$, $\underline{\lim_{n\to\infty}} \mathcal{L}(C_n) \geq \mathcal{L}(C_0)$; whence $\lim_{n\to\infty} \mathcal{L}(C_n) = \mathcal{L}(C_0)$, and A) is proved.

B) follows from A) by Theorem VIII.

VITA

Edward James McShane was born May 10, 1904 in New Orleans, Louisiana. His elementary education was received in the public schools of that city. In 1921 he entered Tulane University of Louisiana, from which he received in 1925 the degrees of Bachelor of Science and Bachelor of Engineering in Mechanical and Electrical Engineering. During the academic years of 1925-26 and 1926-27 he was instructor in mathematics at Tulane University, and also pursued graduate studies in mathematics and physics, receiving the degree of Master of Science in 1927. He was in residence at the University of Chicago during the summer quarter of 1926, from June 1927 to September 1928, and from September 1929 to June 1930; in which periods he studied under Professors Moore, Dickson, Bliss, Lunn, Lane, Graves, Barnard, Bell, Granstein, Compton, and Hoyt. The intervening year, 1928-29, he served as assistant professor of mathematics at the Municipal University of Wichita, Wichita, Kansas.

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